ATME COLLEGE OF ENGINEERING

Department of Electronics and Communication Engineering

NOTES

SEMESTER: IV

SUBJECT: PRINCIPLES OF COMMUNICATION SYSTEM

SUB CODE: BEC402

(ACADEMIC YEAR 2023-24)

Module - 1.

Random Variables and Processes:

* Introduction
* Probability - Conditional Probability.
* Random variables -
* Statistical averages - function of a random variable
  - Moments
* Random Processes -
* Mean, Correlation and covariance functions - Properties of auto-correlation
* Gaussian Process - Cross - Correlation functions
  Gaussian Distribution function.

Text Book 2: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.9. RBT: L1, L2.

Introduction

→ The statistical characterization of random signals is necessary to understand the communication s/m development and is considered as the second pillar of communication theory.

→ A signal is "random" if it is not possible to predict its precise value in advance. Ex: A radio communi-cation system.

→ The received signal in radio communication s/m consists of an information-bearing signal component a random interference component and receiver noise.

Here ① information - bearing signal - voice signal - consists of randomly spaced bursts of energy of random duration.

② The inference component - spurious EM waves produced by other communication s/m.

③ The major source of receiver noise is thermal noise, which is caused by the random motion of electrons in conductors and devices at the front end of the receiver.

→ The precise value of a random signal may be described in terms of its statistical properties such as average power in the random signal, or the average spectral distribution of this power.

→ The mathematical model discipline that deals with the statistical characterization of random signals is Probability Theory.

Random process : It consists of an ensemble (family) of sample functions, each of which varies randomly with time.

Random variable : It is obtained by observing a random process at a fixed instant of time.

2. Probability :

→ Probability Theory is rooted in phenomena that, explicitly or implicitly, can be modeled by an experiment with an outcome that is subjected to chance.

→ Random Experiment : If the experiment is repeated, the outcome can differ because of the influence of an underlying random phenomenon or chance mechanism. Such an experiment is referred to as a random experiment.

Ex: Tossing a fair coin.
The possible outcome of an experiment ~~ofe~~ a trial are "heads" or "tails".

→ There are two approaches to the definition of probability :

Approach 1 : It is based on the relative frequency of occurrence : in 'n' trials of an random experiment, if we expect an event A to occur 'm' times, then we assign the probability $\frac{m}{n}$ to the event A. Ex: Games and engineering situations.

Approach 2 : It is based on set theory and a set of related mathematical axioms. Ex: In situations where experiment is repeatable.

→ In general, when we perform a random expirement, it is natural for us to be aware of the various outcomes that are likely to arise.

→ If ~~the~~ an experiment has K possible outcomes, then for the k^{th} possible outcome we have a point called the sample point, which is denoted as s_k.

Few definitions :

1. The set of all possible outcomes of an experiment is called the sample space, which we denote by 'S'.
2. An event corresponds to either a single sample point or a set of sample points in the space S.
3. A single sample point is called an elementary event.
4. The entire sample space S is called the sure event, & the null set ϕ is called the null or impossible event.
5. Two events are mutually exclusive if the ~~outcomes~~ occurance of one event precludes the occurance of the other event.

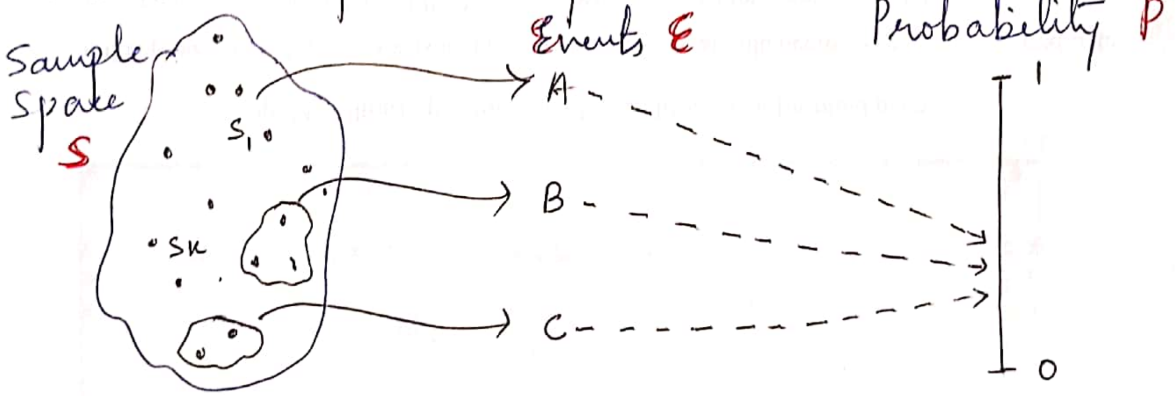
→ The sample space S may be discrete with a countable number of outcomes, such as the outcomes when tossing a die.

→ The sample space S may be continuous, such as the voltage measured at the output of a noise source.

→ A probability measure P is a function that assigns a non-negative number to an event A in the sample space S and satisfies the following 3 properties (axioms):

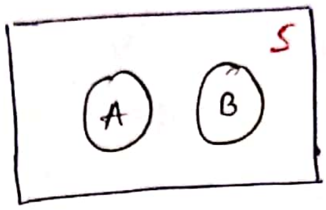
1. $0 \leq P(A) \leq 1$ — ①
2. $P[S] = 1$ — ②
3. If A and B are two mutually exclusive events, then $P[A \cup B] = P[A] + P[B]$. — ③

→ Illustration of the relationship b/w sample space, events and probability is as shown below.



* The sample space S is mapped to events via the random expirement.
* The events may be elementary outcomes of the sample space or larger subsets of the sample space.
* The probability function assigns a value b/w 0 and 1 to each of these events.
* The probability of the union of all events (sure events) is always unity.

→ The three axioms and their relationship to the relative frequency approach may be illustrated by the Venn diagram as shown below.



If we equate P to a measure of the area in the Venn diagram with the total area of S equal to one, then the axioms are simple statements of familiar geometric results regarding area.

→ The following properties of probability measure P may be derived from the above axioms.

1. $P[\bar{A}] = 1 - P[A]$ — (4)
2. When events A & B are not mutually exclusive, then the probability of the union event "A or B" satisfies
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]. \quad — (5)$$
where $P[A \cap B]$ is the probability of the joint event "A and B".
3. If $A_1, A_2, \ldots A_m$ are mutually exclusive that include all possible outcomes of the random experiment, then
$$P[A_1] + P[A_2] + \ldots + P[A_m] = 1 \quad — (6)$$

2.1 Conditional Probability

Let us consider an experiment that involves a pair of events A and B.

Let $P[B/A]$ denotes the probability of event B, given that event A has occured. The probability $P[B/A]$ is called conditional probability of B given A.

Assuming that A has nonzero probability, the conditional probability $P[B/A]$ is defined by

$$P[B/A] = \frac{P[A \cap B]}{P[A]} \quad \text{—— (7)}$$

where $P[A \cap B]$ is the joint probability of A & B.

$$\Rightarrow \quad P[A \cap B] = P[B/A]\, P[A] \quad \text{—— (8)}$$

III^ly

$$P[A \cap B] = P[A/B]\, P[B] \quad \text{—— (9)}$$

Thus,

The joint probability of two events may be expressed as the product of the conditional probability of one event given the other, and the elementary probability of the other.

Now, Let substitute eq.ⁿ (9) in eq.ⁿ (7), we get

$$P[B/A] = \frac{P[A/B]\, P[B]}{P[A]} \quad \text{—— (10)}$$

This relation is called Baye's rule.

This situation exist where the conditional probability $P[A/B]$ & the probabilities $P[A]$ & $P[B]$ are easily determined directly, but conditional probability $P[B/A]$ is desired.

Suppose,

The conditional probability $P[B/A]$ is simply equal to $P[B]$ i.e

$$P[B/A] = P[B] \quad \text{—— (11)}$$

Under this condition, the probability of occurance of the joint event $A \cap B$ is equal to the product of the elementary probabilities of the events A and B,

i.e $P[A \cap B] = P[A]\, P[B]$ [Substituting — (12) eqn (11) in eqn (8)]

so that

$$P[A|B] = P[A] \quad \text{— (13)}$$

i.e, the conditional probability of event A, assuming the occurance of event B, is simply equal to the elementary probability of event A.

Thus, we see that in this case a knowledge of the occurrance of one event tells us no more about the probability of occurrance of the other event that we knew without that knowledge.

Event A & B that satisfy this condition are said to be statistically independent.

Example 1 : Binary Symmetric Channel.

→ Consider a discrete memoryless channel used to transmit binary data. [discrete - binary values, memoryless - o/p depends only on i/p at particular time]

→ ~~Due~~ to the presence of noise, errors are made at in the received binary data stream, i.e symbol 1 is sent, but symbol 0 is received or vice versa.

→ The channel is assumed to be symmetric, which means that the probability of receiving symbol 1 when symbol 0 is sent is the same as the probability of receiving symbol 0 when symbol 1 is sent.

→ To describe the probabilistic nature of this channel fully, we need two sets of probabilities.

① The a priori probabilities of sending binary symbols 0 and 1, they are

$$P[A_0] = p_0 \quad \& \quad P[A_1] = p_1$$

where A_0 & A_1 → events of transmitting symbols 0 & 1, respt.

Note: $p_0 + p_1 = 1$

② The conditional probabilities of error, they are

$$P[B_1 | A_0] = P[B_0 / A_1] = p.$$

where B_0 & B_1 → events of receiving symbols 0 & 1 respt.

$P[B_1 | A_0]$ — probability of receiving symbol 1, given that symbol 0 is sent.

$P[B_0 / A_1]$ — probability of receiving symbol 0, given that symbol 1 is sent.

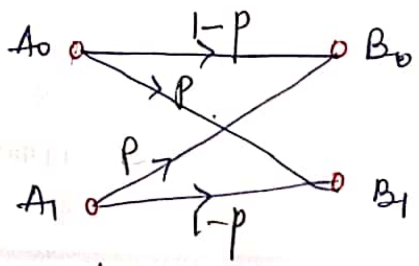
→ Here the requirement is to determine the a posteriori probabilities i.e $P[A_0/B_0]$ & $P[A_1 | B_1]$. Both these conditional probabilities refers to events that are observed "after the fact", hence the name "a posteriori" probabilities.

→ Since B_0 & B_1 are mutually exclusive, from axioms we can write

$$P[B_0/A_0] + P[B_1/A_0] = 1$$

$$\Rightarrow \quad P[B_0/A_0] = 1 - p \qquad [\because P[B_1/A_0] = p]$$

$|||^{ly}$ $P[B_1 | A_1] = 1 - p.$



Using these eqn., let us draw transition probability diagram as shown below, which represents symmetric binary channel.

→ Using this diagram, let deduce the following results.

① The probability of receiving symbol 0 is given by

$$P[B_0] = P[B_0/A_0]\,P[A_0] + P[B_0/A_1]\,P[A_1]$$
$$= (1-p)p_0 + pp_1$$

② The probability of receiving symbol 1 is given by

$$P[B_1] = P[B_1/A_0]\,P[A_0] + P[B_1/A_1]\,P[A_1]$$
$$= pp_0 + (1-p)p_1 \quad \text{— (14)}$$

Therefore, applying Bayes' rule, we obtain

$$P[A_0/B_0] = \frac{P[B_0/A_0]\,P[A_0]}{P[B_0]}$$

$$\Rightarrow \boxed{P[A_0/B_0] = \frac{(1-p)p_0}{(1-p)p_0 + pp_1}}$$

$$P[A_1|B_1] = \frac{P[B_1/A_1]\,P[A_1]}{P[B_1]}$$

$$\Rightarrow \boxed{P[A_1|B_1] = \frac{(1-p)p_1}{pp_0 + (1-p)p_1}}$$

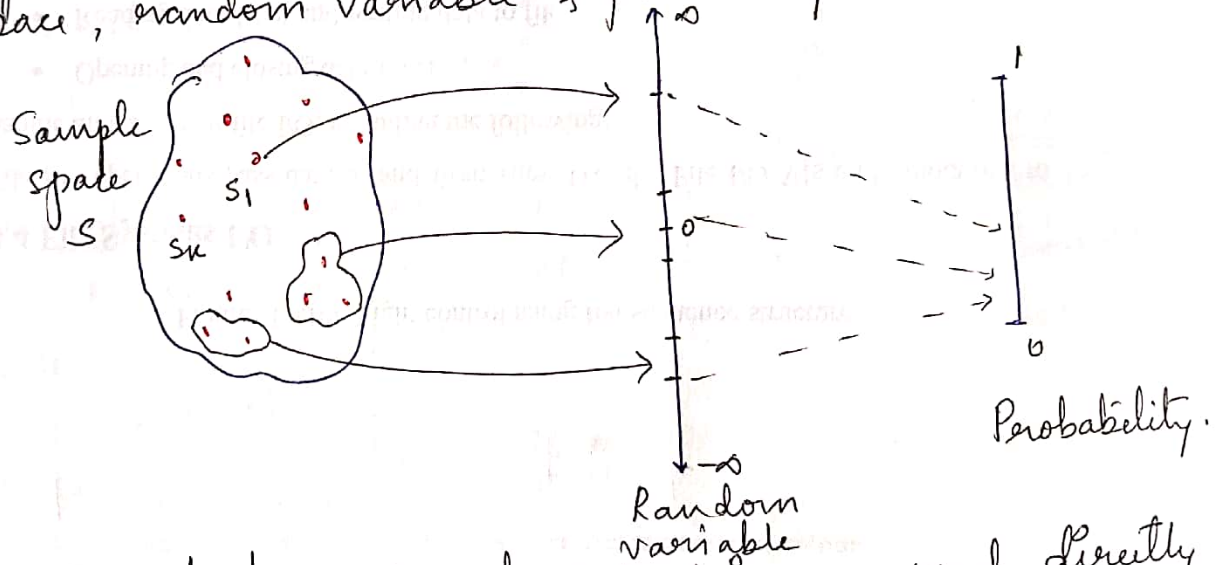
The desired results are a posteriori probabilities.

5.3 Random Variables

→ A function whose domain is a sample space and whose range is a set of real numbers is called a random variable of the expirement.
i.e for events in E, a random variable assigns a subset of the real line.
Thus, if the outcome of the expirement is 's', we denote the random variable as $X(s)$ or simply X.
We denote a particular outcome of a random expirement by x, i.e $X(s_k) = x$

Note: There may be more than one random variable associated with the same random expirement.

→ Fig below illustrates the relationship b/w sample space, random variable & probability.



Here subsets of samplespace being mapped directly to a subset of the real line, as random variable.
The probability function applies to this random variable is exactly the same manner that it applies to the underlying events.

→ Random variable can be 2 types
1. Discrete RV & 2. Continous RV.

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→ Discrete RV: A random variable that can take only a finite number of values, such as in the coin-tossing experiment.

→ Continuous RV: A random variable that can take a range of real values. Ex: amplitude of a noise voltage at a particular instant in time, whose value vary b/w plus & minus infinity.

Note: RV may be complexed valued, but a complex-valued RV may always be treated as a vector of 2 real-valued RV.

→ Probabilistic Description of RV

* Let consider the RV X & the probability of the event $X \le x$, denoted as $P[X \le x]$.

To simply our ~~do~~ notations, we write

$$F_X(x) = P[X \le x] \quad \text{—— (15)}$$

This function $F_X(x)$ is called the cumulative distribution function (CDF), which is a function of x [not the RV X]. Also called as Distribution f^n.

Note: For any point x, the CDF $F_X(x)$ expresses a probability.

Properties of CDF:

1. The distribution f^n $F_X(x)$ is bounded b/w 0 & 1.
2. The distribution f^n $F_X(x)$ is a monotone-non decreasing f^n of x, i.e

$$F_X(x_1) \le F_X(x_2) \text{ if } x_1 < x_2 . \quad \text{—— (16)}$$

* The distribution fun of a RV always exist. If the distribution f^n is continously differentiable than an alternate discription of the probability of the RV is useful.

→ The derivative of the distribution function is called the probability density function (PDF) of the RV X, defined as

$$f_X(x) = \frac{d}{dx} F_X(x) \quad \text{——} (17)$$

→ The density f^n indicates the probability of the event $x_1 < X \leq x_2$ equals

$$P[x_1 < X \leq x_2] = P[X \leq x_2] - P[X \leq x_1]$$
$$= F_X(x_2) - F_X(x_1)$$
$$= \int_{x_1}^{x_2} f_X(x)\, dx \quad \text{——} (18)$$

i.e the probability of an interval is therefore the area under the PDF in that interval.

Suppose $x_1 = -\infty$, then $F_X(x)$ is defined in terms of PDF as

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi \quad \text{——} (19)$$

Since $F_X(\infty) = 1$, corresponding to the probability of a certain event, and
$F_X(-\infty) = 0$, corresponding to the probability of an impossible event, we can find from eq^n (18) that

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \quad \text{——} (20)$$

Note: A PDF must always be a non-negative f^n and with a total area of 1.

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Example 2 : Uniform Distribution

A random variable X is said to be uniformly distributed over the interval (a, b) if the

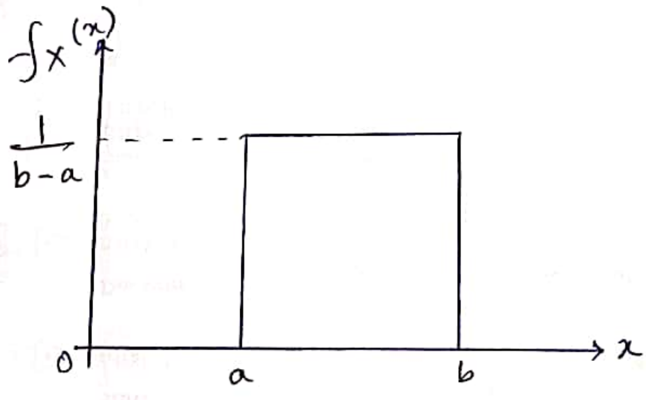
PDF is :

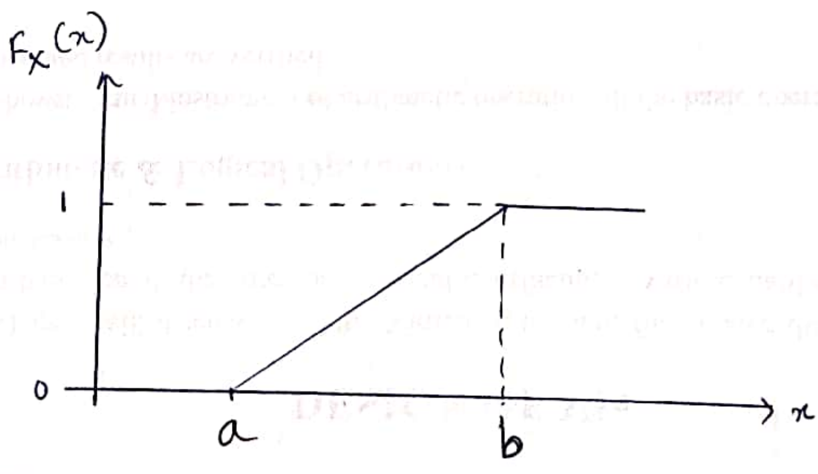
$$f_X(x) = \begin{cases} 0, & x \leq a \\ \frac{1}{b-a}, & < x < \\ 0, & x > b \end{cases} \qquad \text{—— (21)}$$

CDF is :

$$F_X(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 0, & x > b \end{cases} \qquad \text{—— (22)}$$

plots of the PDF & CDF of the uniformly distributed RV X is as shown below





4. Statistical Averages.

After discussing probability, now let us determine the average behaviour of the outcomes arising in random expirements.

Expected Value or Mean:

The expected value or Mean of the RV is defined as

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx. \quad \text{--- (23)}$$

where E denotes the statistical expectation operator i.e the mean μ_X locates the center of gravity of the area under the probability density curve of RV.

4.1. Function of a RV.

Let X denote a RV, let g(X) denote a real-valued function defined on real line. Let denote $Y = g(X)$ --- (24) Now, let find E[Y] using the standard formula.

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy \quad \text{--- (25)}$$

$$\Rightarrow \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \quad \text{--- (26)}.$$

Example 3: Cosinusoidal RV

Let $Y = g(X) = \cos(X)$, where X is RV uniformly distributed in the interval $(-\pi, \pi)$.

i.e
$$f_X(x) = \begin{cases} \frac{1}{2\pi} & -\pi < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

Thus E[Y] is

$$E[Y] = \int_{-\pi}^{\pi} (\cos x)\left(\frac{1}{2\pi}\right) dx = \frac{-1}{2\pi} \sin x \Big|_{-\pi}^{\pi} = 0$$

4.2. Moments

→ For the speical case of $g(x) = x^n$, using eq.ⁿ (26)
we can obtain the n^{th} moment of the probability distribution of the RV, X

i.e
$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx \qquad (27)$$

Let put n=1, we get expected value of RV &
if we put n=2, we get the mean-square value of X.

i.e
$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \qquad (28)$$

→ N^{th} central moment : It is the moments of the difference b/w a RV X and its mean μ_X.

Thus
$$E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x)\, dx \qquad (29)$$

Now, for n=1, the central moment is, of course, zero, whereas for n=2 the second central moment is referred as the variance of the RV X, written as

$$Var[X] = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx \qquad (30)$$

Variance, Var[X] is also denoted as σ_X^2.

→ Standard deviation : The square root of the variance is called the standard deviation of the RV X.

→ Relation b/w Variance & mean square value :

$$\sigma_X^2 = E[X^2 - 2\mu_X X + \mu_X^2]$$
$$= E[X^2] - 2\mu_X E[X] + \mu_X^2 \qquad \text{[using linearity property]}$$
$$\Rightarrow \boxed{\sigma_X^2 = E[X^2] - \mu_X^2} \qquad (31)$$

Note : when $\mu_X = 0$, $\boxed{\sigma_X^2 = E[X^2]}$

5. Random Processes.

In describing random signals, each sample point is in sample space is a function of time.
The sample space or ensemble comprised of functions of time is called a Random or Stochastic process.

Consider a random experiment specified by the outcomes s from some sample space S, by the events defined on the sample space S, and by the probabilities of these events.
Suppose that we assign to each sample point s a function of time in accordance to the rule:

$$X(t, s) \quad -T \leq t \leq T \qquad (32)$$

where observation interval = 2T.

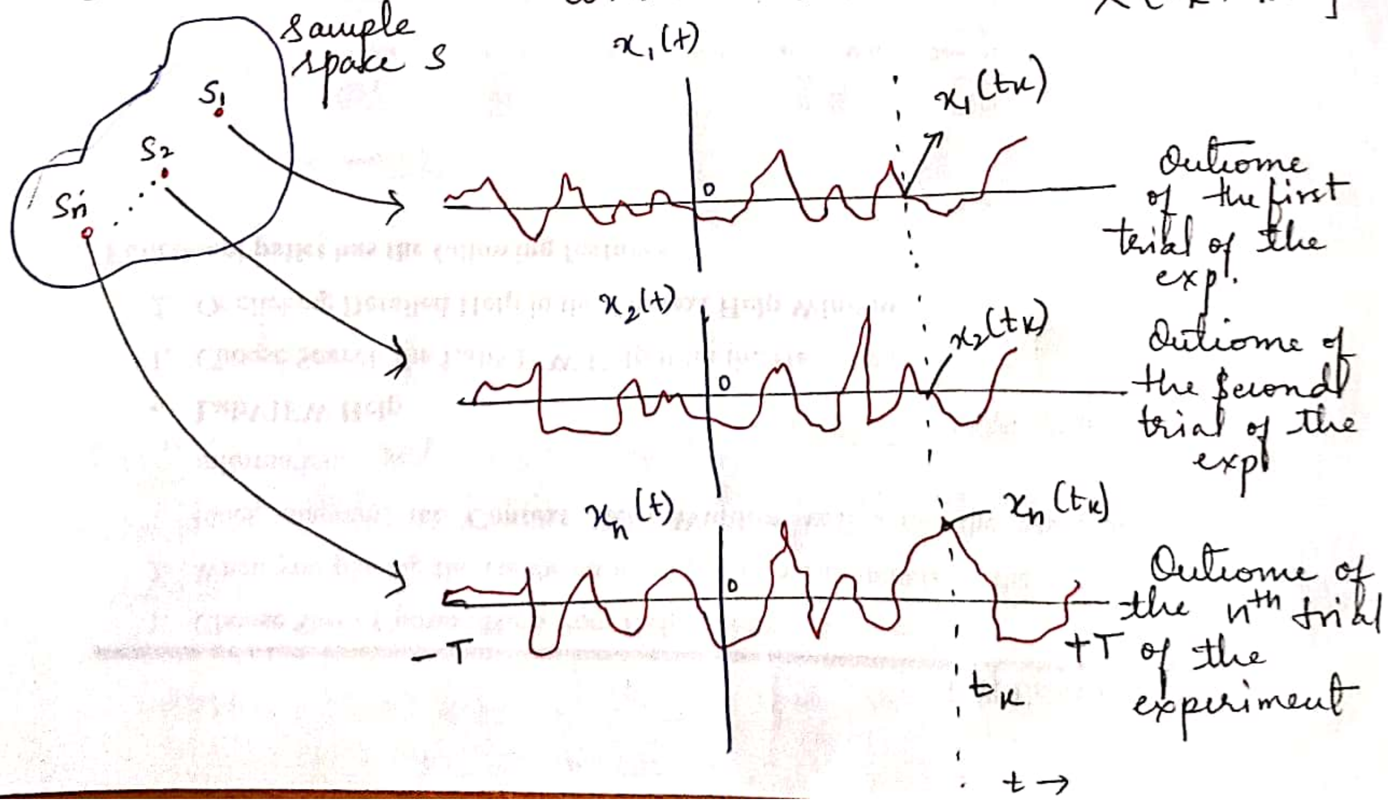
For fixed sample point s_j, it is denoted as

$$x_j(t) = X(t, s_j) \qquad (33)$$

fig below illustrates the set of sample functions $\{x_j(t) \mid j = 1, 2, \ldots n\}$ & can be denoted as, at fixed time t_k, it

$$\{x_1(t_k), x_2(t_k), \ldots x_n(t_k)\} = \{X(t_k, s_1), X(t_k, s_2), \ldots X(t_k, s_n)\}$$

constitutes a RV.



$X(t,s)$ denotes random process. simplified notation $X(t)$.

Thus,
A random process $X(t)$ as an ensemble of time functions together with a probability rule that assigns a probability to any meaningful event associated with an observation of one of the sample functions of the random process.

Distinguish between a random variable and a random process as follows.
→ For a random variable, the outcome of a random experiment is mapped into a number.
→ For a random process, the outcome of a random experiment is mapped into a waveform that is a function of time.

6. Mean, Correlation and Covariance functions

Mean: For a random process $X(t)$, the mean is defined as the expectation of the RV obtained by observing the process at some time t, given by

$$\mu_X(t) = E[X(t)]$$

$$= \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx \quad \text{—— (34)}$$

where $f_{X(t)}(x) \rightarrow$ PDF of the process at time t.

Note: If ~~RP~~ $X(t_1)$ & $X(t_2)$ are two RP and are said to be stationary to first order if it satisfies $f_{X(t_1)}(x) = f_{X(t_2)}(x)$. for all t_1 & t_2. —— (35)

Also $\mu_X(t) = \mu_X$ for all t —— (36)

$\sigma_X^2(t) = \text{const}$.

Autocorrelation function: It is defined as the expectation of the product of two random processes.

i.e $\quad R_X(t_1, t_2) = E[X(t_1)X(t_2)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2 \quad \text{--- (37)}$$

where $f_{X(t_1),X(t_2)}(x_1, x_2)$ is the joint PDF of the RP $X(t_1)$ & $X(t_2)$.

Note: A random process $X(t)$ is said to be stationary to second order if the joint density function $f_{X(t_1),X(t_2)}(x_1, x_2)$ depends only on the difference b/w the observation times t_1 & t_2.

Similarly, the auto-correlation fn. of a stationary (second order) RP depends only on the time difference $t_2 - t_1$ as

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \text{ for all } t_1 \,\&\, t_2. \quad \text{--- (38)}$$

Autocovariance function:

The autocovariance function of a stationary random process $X(t)$ is written as

$$C_X(t_1, t_2) = E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)]$$

$$= R_X(t_2 - t_1) - \mu_X^2. \quad \text{--- (39)}$$

Note: Two important points that should be carefully noted.

1. The mean & autocorrelation fn. only provide a partial description of the distribution of a RP $X(t)$.
2. The conditions defined in eqn. 36 & 38 are not sufficient to guarantee that the RP is stationary.

5.1 Properties of the autocorrelation function.

For convenience, the autocorrelation f^n of a stationary process $X(t)$ is redefined as

$$R_X(\tau) = E[X(t+\tau)\, X(t)] \quad \text{for all } t$$

Properties :

1. The mean square value of the process can be obtained from $R_X(\tau)$ by substituting $\tau = 0$ in above eq^n, as shown by

$$R_X(0) = E[X^2(t)]$$

2. The autocorrelation function $R_X(\tau)$ is an even function of τ

i.e $\quad R_X(\tau) = R_X(-\tau)$

3. The autocorrelation function $R_X(\tau)$ has its maximum magnitude at $\tau = 0$,

i.e $\quad |R_X(\tau)| \leq R_X(0)$

Proof: Consider the nonnegative quality

$$E[(X(t+\tau) \pm X(t))^2] \geq 0$$

Expanding & taking individual expectations, we get

$$E[X^2(t+\tau)] \pm 2E[X(t+\tau)\,X(t)] + E[X^2(t)] \geq 0$$

$$\Rightarrow \quad 2R_X(0) \pm 2R_X(\tau) \geq 0$$

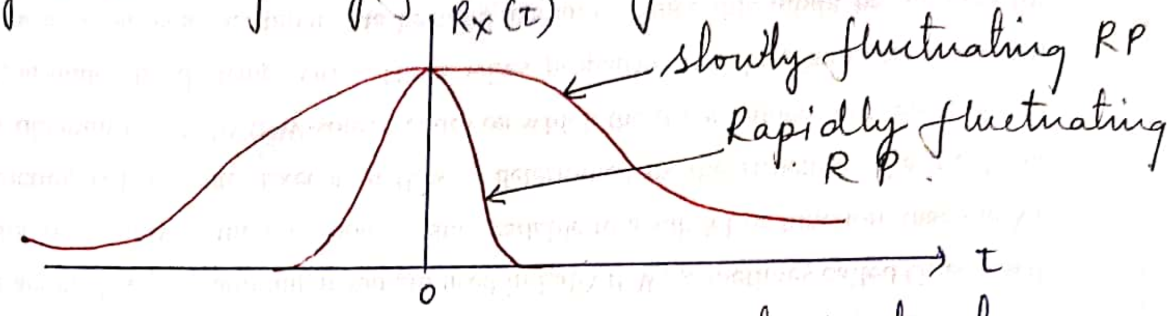
Equivalently, we can write

$$-R_X(0) \leq R_X(\tau) \leq R_X(0)$$

$$\Rightarrow \quad \boxed{|R_X(\tau)| \leq R_X(0)}$$

→ The physical significance of the autocorrelation function $R_X(\tau)$ is that it provides a means of describing the "interdependence" of 2 RV obtained by observing a RP $X(t)$ at time τ sec apart.

The fig below illustrates the autocorrelation f^n's of slowly and rapidly fluctuating Random processes.



Note: The decrease may be characterized by a decorrelation time τ_0, such that for $\tau > \tau_0$, the magnitude of the autocorrelation f^n. $R_X(\tau)$ remains below some prescribed value.

Example 4: Sinusoidal Signal with Random Phase.

Consider a sinusoidal signal with random phase defined by $X(t) = A\cos(2\pi f_c t + \Theta)$

where A & f_c are const &
Θ is a random variable that is uniformly distributed over the interval $(-\pi, \pi)$

i.e
$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi < \theta < \pi \\ 0, & \text{otherwise/elsewhere.} \end{cases}$$

The autocorrelation f^n of $X(t)$ is obtained as

$$R_X(\tau) = E[X(t+\tau)X(t)]$$
$$= E[(A\cos(2\pi f_c(t+\tau)+\Theta))(A\cos(2\pi f_c t+\Theta))]$$
$$= E[A^2\cos(2\pi f_c t + 2\pi f_c\tau + \Theta)\cos(2\pi f_c t + \Theta)]$$
$$= \frac{A^2}{2}E[\cos(4\pi f_c t + 2\pi f_c\tau + 2\Theta)] + \frac{A^2}{2}E[\cos 2\pi f_c\tau]$$

since WKT

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

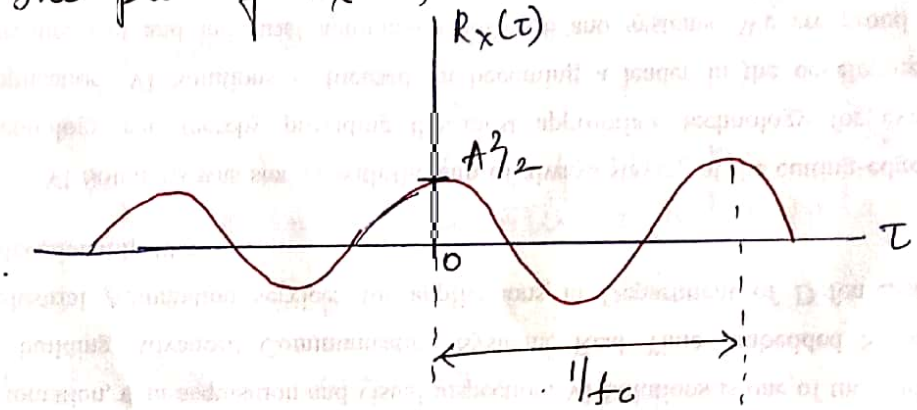
Applying expectation to each term separately, we get

$$R_X(\tau) = \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi f_c t + 2\pi f_c\tau + 2\theta)\,d\theta + \frac{A^2}{2}\cos 2\pi f_c\tau$$

Here, the first term integrates to zero, we get

$$R_X(\tau) = \frac{A^2}{2}\cos 2\pi f_c\tau$$

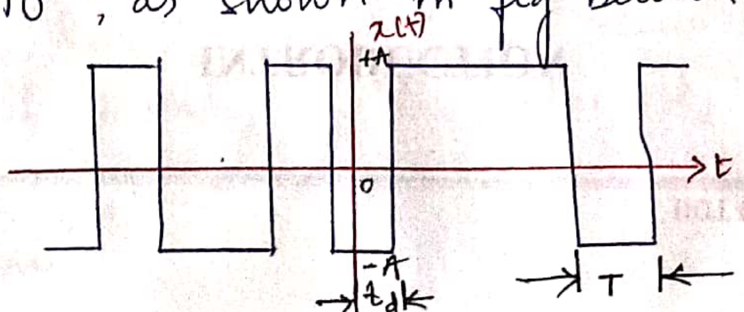
The plot of $R_X(\tau)$ is as shown below.



Therefore, the autocorrelation function of a sinusoidal signal with random phase is another sinusoidal at the same frequency in the 'τ domain' rather than the original time domain.

Example 5: Random Binary Signal

Consider a sample function $x(t)$ of a process $X(t)$ consisting of a Random sequence of binary symbols 1 and 0, as shown in fig below.



Let consider the following assumptions:

1. The symbol 1 is represented by pulse of amplitude $(+A)$ & symbol 0 is represented by pulse of amplitude $(-A)$.

2. The pulses are not synchronized, so starting time t_d is likely to lie b/w 0 to T. Also let t_d be the sample value of RV T_d & PDF is defined as

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T} & 0 \le t_d \le T \\ 0 & \text{elsewhere} \end{cases}$$

3. During any time interval $(n-1)T < t - t_d < nT$, where n is an integer, then the presence of 0 & 1 is equally likely in any one interval is independent of all other intervals.

i.e $E[X(t)] = 0$ for all t

Let find Autocorrelation.

Consider $X(t_k)$ & $X(t_i)$ are two RP at times t_k & t_i respt.

We can analyze this, ~~consid~~ using two below cases

Case 1: when $|t_k - t_i| > T$

This mean that RP $X(t_k)$ & $X(t_i)$ occur in different pulse intervals & are therefore independent

$\therefore$ $E[X(t_k)\,X(t_i)] = E[X(t_k)]\,E[X(t_i)] = 0$, for $|t_k - t_i| > T$

Case 2: when $|t_k - t_i| < T$ with $t_k = 0$ & $t_i < t_k$

Here, RP $X(t_k)$ & $X(t_i)$ occur in same pulse interval if & only if the delay t_d satisfies the condition $t_d < T - |t_k - t_i|$.

~~Thus~~ ~~tow~~

Thus conditional expectation

$$E[X(t_k)X(t_i)/t_d] = \begin{cases} A^2, & t_d < T - |t_k - t_i| \\ 0, & \text{elsewhere} \end{cases}$$

Averaging this result over all possible values of t_d, we get

$$E[X(t_k)X(t_i)] = \int_0^{T-|t_k - t_i|} A^2 f_{T_d}(t_d)\, dt_d$$

$$= \int_0^{T-|t_k - t_i|} \frac{A^2}{T}\, dt_d$$

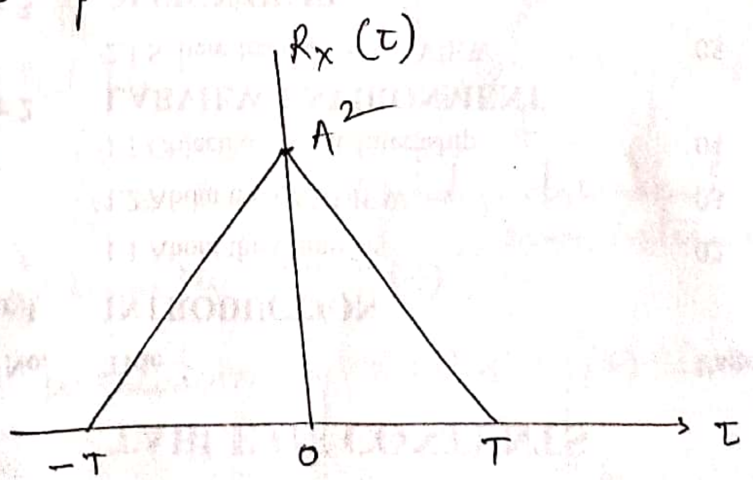
$$= A^2\left(1 - \frac{|t_k - t_i|}{T}\right), \quad |t_k - t_i| < T$$

Thus considering $\tau = t_k - t_i$,

Auto correlation fn. is given as

$$R_X(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ 0 & |\tau| \geq T \end{cases}$$

This is plotted as shown below



7. Cross Correlation functions

Let $X(t)$ and $Y(t)$ be two random processes with autocorrelation functions $R_X(t,u)$ & $R_Y(t,u)$ respt. The cross-correlation function of $X(t)$ & $Y(t)$ is defined by $R_{XY}(t,u) = E[X(t)Y(u)]$

Note: If the R.P $X(t)$ and $Y(t)$ are each wide-sense stationary the cross-correlation may be written as

$$R_{XY}(t,u) = R_{XY}(\tau) \quad \text{where } (t-u)=\tau$$

Properties:

1. The cross correlation function is not generally an even function of τ.
2. It doesnot have a maximum at the origin,
3. It obeys symmetric relationship as follows
$$R_{XY}(\tau) = R_{XY}(-\tau).$$

Example : 6 : Quadrature - Modulated Processes

Consider a pair of quadrature-modulated processes $X_1(t)$ & $X_2(t)$ that are related to wide-sense stationary process $X(t)$ as follows:

$$X_1(t) = X(t)\cos(2\pi f_c t + \Theta)$$
$$X_2(t) = X(t)\sin(2\pi f_c t + \Theta)$$

where $f_c \rightarrow$ carrier freq &
$\Theta \rightarrow$ R.V (uniformly distributed over interval $(0, 2\pi)$)

Here Θ is independent of $X(t)$.

Thus cross correlation of $X_1(t)$ & $X_2(t)$ is given as

$$R_{12}(\tau) = E[X_1(t)X_2(t-\tau)]$$
$$= E[X(t)\cos(2\pi f_c t + \Theta).X(t-\tau)\sin(2\pi f_c t - 2\pi f_c \tau + \Theta)]$$

$$\Rightarrow R_{12}(\tau) = E[x(t)\,x(t-\tau)]\,E[\cos(2\pi f_c t+\Theta)\sin(2\pi f_c t-2\pi f_c\tau+\Theta)]$$

WKT $\cos A \sin B = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$

$$\Rightarrow R_{12}(\tau) = \frac{1}{2}R_x(\tau)\,E[\sin(4\pi f_c t-2\pi f_c\tau+2\Theta) - \sin(2\pi f_c\tau)]$$

$$\boxed{R_{12}(\tau) = -\frac{1}{2}R_x(\tau)\sin(2\pi f_c\tau)}$$

Note:- At $\tau=0$, $\sin 2\pi f_c\tau$ is zero

$\therefore R_{12}(0) = E[x_1(t)\,x_2(t)]$

$= 0$

i.e The random variables obtained by simultaneously observing the quadrature-modulated processes $X_1(t)$ & $X_2(t)$ at some fixed value of time t are orthogonal to each other.

8. Gaussian Process

An important family of random process is Gaussian Processes.

Let us consider a ~~stor~~ random process $X(t)$ for an interval that starts at time $t=0$ and lasts until $t=T$. Suppose that we weight the random process $X(t)$ by some function $g(t)$ and then integrate the product $g(t)\,X(t)$ over the observational interval, thereby obtaining a random variable Y defined by

$$Y = \int_0^T g(t)\,X(t)\,dt$$

where Y is a linear function of $X(t)$.

If the weighting function $g(t)$ is such that the mean-square value of the random variable Y is finite, and if the random variable Y is a Gaussian-distributed random variable for every $g(t)$ in this class of functions, then the process $X(t)$ is said to be a Gaussian process.

The random variable Y has a Gaussian distribution if its probability density function has the form

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]$$

where μ_Y is the mean &

σ_Y^2 is the variance of the random variable of Y.

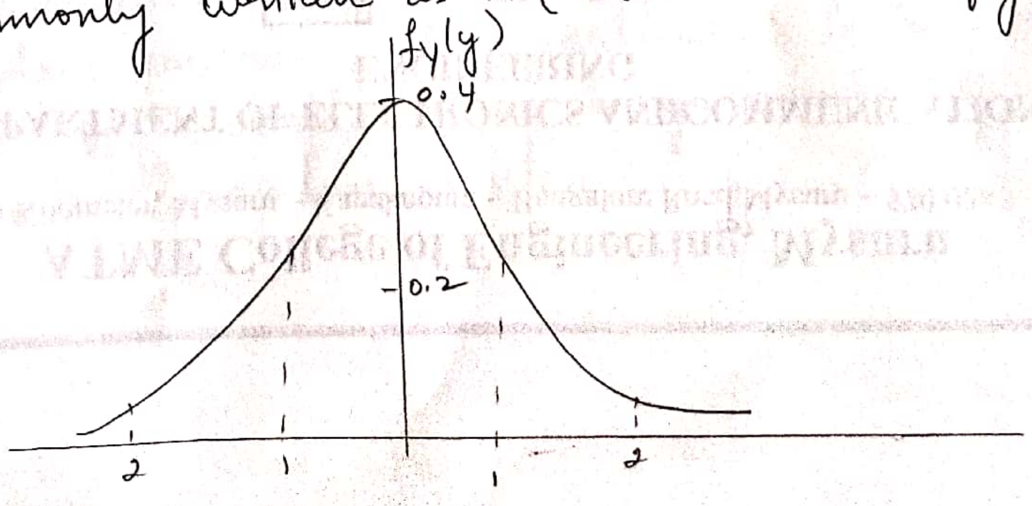
~~The random variable Y has a Gaussian distribution if its probability density f^n has the form~~

$\cancel{f_Y(y)}$

The random variable Y has a Gaussian distribut^n if & is normalized to have a mean μ_Y of zero & a variance σ_Y^2 of one, as shown by

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

Such a normalized Gaussian distribution is commonly written as $N(0,1)$ as shown in fig below.



Central limit theorem: It states the probability distribution of V_N approaches a normalized Gaussian distribution $\mathcal{N}(0,1)$ in the limit as N approaches infinity.

Proof: Let X_i, $i = 1, 2, \ldots, N$ be a set of random variables that satisfies the following requirements

1. The X_i are statistically independent.
2. The X_i have the same probability distribution with mean μ_X and variance σ_X^2.

If above conditions are satisfied, such variables constitute a set of independently & identically distributed RV (i.i.d. RVs)

Let these random variables be normalized as follows;

$$Y_i = \frac{1}{\sigma_X}(X_i - \mu_X), \quad i = 1, 2, \ldots N$$

so that, we have $E[Y_i] = 0$ &

$Var[Y_i] = 1$

Thus we define RV, $\boxed{V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i}$,

Thus, the central limit theorem provides the mathematical justification for using a gaussian process as a model for a large number of different physical phenomenon in which the observed random variable, at a particular instant of time, is the result of a large number of individual random events.

Properties of a Gaussian Process

Property 1: If a Gaussian process $X(t)$ is applied to a stable linear filter, then the random process $Y(t)$ developed at the output of the filter is also Gaussian, i.e $Y(t) = \int_0^T h(t-\tau)\, X(\tau)\, d\tau \quad 0 \le t < \infty$

Property 2: Consider the sets of random variables or samples $X(t_1), X(t_2), \ldots X(t_n)$, obtained by observing a random process $X(t)$ at times $t_1, t_2, t_3, \ldots t_n$. If the process $X(t)$ is Gaussian, then this set of random variables is jointly Gaussian for any n, with their n-fold joint PDF being completely determined by specifying the set of means

$$\mu_{X(t_i)} = E[X(t_i)] \quad i = 1, 2, \ldots n$$

& a set of autocovariance functions

$$C_X(t_k, t_i) = E[(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})]$$

$$k, i = 1, 2, \ldots n$$

Property 3: If a Gaussian process is wide-sense stationary, then the process is also stationary in the strict sense.

Property 4: If a random variables $X(t_1), X(t_2), \ldots X(t_n)$ obtained by sampling a Gaussian process $X(t)$ at times $t_1, t_2, \ldots t_n$ are uncorrelated, i.e

$$E[(X(t_k) - \mu_{X(k)})(X(t_i) - \mu_{X(t_i)})] = 0 \quad i \ne k$$

then these random variables are statistically independent.